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LEADING LARGE N MODIFICATION OF QCD_2 ON A CYLINDER BY DYNAMICAL FERMIONS

Avinash Dhar*, Gautam Mandal and Spenta R. Wadia
Tata Institute of Fundamental Research
Homi Bhabha Road, Bombay 400 005, INDIA

ABSTRACT

We consider 2-dimensional QCD on a cylinder, where space is a circle. We find the ground state of the system in case of massless quarks in a $1/N$ expansion. We find that coupling to fermions nontrivially modifies the large N saddle point of the gauge theory due to the phenomenon of 'decompactification' of eigenvalues of the gauge field. We calculate the vacuum energy and the vacuum expectation value of the Wilson loop operator both of which show a nontrivial dependence on the number of quarks flavours at the leading order in $1/N$.

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e-mail: adhar, mandal, wadia@theory.tifr.res.in
*Address after October 1, 1994: Theory Division, CERN, CH-1211, Geneve 23, Switzerland.

0 Introduction and Summary

The large N expansion provides a valuable tool for obtaining qualitative insight into gauge theories [1]. The $1/N$ expansion in gauge theories is based on the discovery by 'tHooft [2] that for $SU(N)$ gauge theories with fundamental fermions the N dependence of a typical vacuum diagram $E_{H,L}$ for large N is

$$E_{H,L} \propto N^2 (\frac{1}{N^2})^H (\frac{1}{N})^L \tag{0.1}$$

where H is the number of handles and L the number of holes (fermion loops) in the diagram. It is easy to see that the dominant contributions to the vacuum energy occurs when $H = L = 0$ and that it is proportional to N^2. This simply reflects the fact that there are N^2 gluons in the theory. The fermionic contribution to the vacuum energy, on the other hand, must have at least one fermion loop, and the leading contribution is $E_{0,1} \propto N$. This, again, reflects the fact that there are N fermions in a fundamental representation. The N^2-dependence of the vacuum energy has been explicitly calculated in soluble large N matrix models [3] which, like gauge theories, also possess N^2 degrees of freedom. On the other hand, two-dimensional QCD (QCD_2) on the plane, which is another soluble model at large N [4], possesses no dynamical gauge degrees of freedom and has vacuum energy $\propto N$ coming from the N fermions. See also the more recent works on pure QCD_2 [5] and on QCD_2 with fermions [6, 7].

In the present paper we continue [7] our study of large N QCD_2 on a cylinder. Here space is a circle and hence the gauge field does not decouple. After fixing gauge appropriately the gauge field can be described by the N eigenvalues of A_1 which satisfy fermi statistics. The standard wisdom, based on Eqn. (0.1) and the discussion in the last paragraph, would suggest the following two-step procedure for solving the theory at large N — (1) to solve the pure gluon theory first (the large N saddle point of this is easy to determine and is described by a constant density of eigenvalues; see remarks before (3.47) below) and (2) treat the fermion dynamics subsequently as fluctuations in the fixed external gauge field background determined in step (1). In this paper we explicitly show that such a procedure is incorrect. The reason for this, basically, is that in presence of quarks there are gauge-invariant operators which (unlike the Wilson loop operator) are *not periodic* functions of the eigenvalues and this effectively leads to a noncompact range of the

eigenvalues unlike in the pure Yang-Mills theory. This is the phenomenon of 'decompactification' described in great detail in [7]. It is clear that a constant density of eigenvalues is not normalizable in a noncompact domain. Indeed we find that the quarks lead to a harmonic oscillator potential for the eigenvalues, resulting in expressions for physical quantities that are nontrivially different from those of pure Yang-Mills theory at the leading N order. We present our main results below.

The vacuum energy in our theory is given by (see Eqn. (3.43) below)

$$\bar{E}_0 = N^2[\sqrt{n_f}\frac{\bar{g}}{2\pi} + o(1/N)] \tag{0.2}$$

where n_f is the number of quark flavours and $\bar{g} = g\sqrt{N}$ as usual denotes the scaled coupling constant. The vacuum expectation value of the Wilson loop operator $W_m = (1/N)\mathrm{Tr}U^m,\ U = \exp[i\int_0^L A_1 dx]$ is given by (see Eqn. (3.46))

$$\langle W_m \rangle = 2(-1)^m(1+\frac{\partial^2}{\partial {x_m}^2})J_0(x_m)+o(1/N), \quad x_m \equiv 2\pi^{3/4}\alpha\frac{m}{n_f^{1/4}},\ \alpha = \sqrt{\bar{g}\bar{L}/2\pi} \tag{0.3}$$

In the above $\bar{E}_0 \equiv E_0\sqrt{N}$ and $\bar{L} \equiv L\sqrt{N}$ denote an additional N-scaling necessary in order to have a well-defined large N limit in our theory. The expression for the eigenvalue density is given by Eqn. (3.44).

Note the nontrivial dependence of the above expressions on the number of quark flavours which clearly shows that adding quarks changes the leading large N result. For comparison, note that the vacuum expectation value of the Wilson loop operator in pure Yang-Mills theory in two dimensions is (see (3.47) below) $\langle W_m \rangle_{YM} = \delta_{m,0} + o(1/N)$.

This paper is organized as follows. In Sec. 1 we write down the largrangian, the path integral and the hamiltonian for QCD on a two-dimensional cylinder. We follow the notation and results of [7]. In Sec. 2 we find the ground state of the theory in the $1/N$ expansion and reduce the calculation of the vacuum energy to a problem of N interacting fermions in a harmonic oscillator external potential. The solution of the latter problem is presented in Sec. 3 in the $1/N$ expansion which allows us to calculate the vacuum energy of the full system. We find in the process that we need to scale both the radius of the cylinder and the time in a certain way (see Eqns. (3.29) and (3.31)) to have a well-defined large N limit. Besides the vacuum energy we

also calculate the vacuum expectation value of the Wilson loop operator. In Sec. 4 we conclude with some comments about possible relevance to four dimensions.

1 The Action and the Hamiltonian

We consider the gauge group $U(N)$. As usual we denote the gauge fields, which are hermitian matrices, by A_μ^{ab} where $\mu = 0, 1$ is the Lorentz index and $a, b = 1, 2, \dots, N$ are colour indices. The fermions are denoted by $\psi^a_{i\alpha}$ where $i = 1, 2, \dots, n_f$ is the flavour index and $\alpha = \pm 1$ is the dirac index. We consider one space and one time dimension where the space dimension is a circle of length L.

The theory is described by the following path-integral

$$\begin{aligned} &Z = \int \mathcal{D}A_0(x,t)\mathcal{D}A_1(x,t)\mathcal{D}\psi(x,t)\mathcal{D}\psi^\dagger(x,t)\exp[iS(A_0, A_1, \psi, \psi^\dagger)] \\ &S = \int_0^T dt \int_0^L dx(- \tfrac{1}{4}\mathrm{tr}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\gamma^\mu(iD_\mu)\psi - m\bar{\psi}\psi) \\ &F_{01} \equiv E = \partial_t A_1 - \partial_x A_0 + ig[A_0, A_1], \quad D_\mu = \partial_\mu + igA_\mu \end{aligned} \tag{1.1}$$

In this paper we will only consider the case of massless quarks, $m = 0$.

Let us fix the gauge

$$A_1(x,t) = \mathrm{Diag}[\lambda_a(t)] \tag{1.2}$$

The gauge-fixed path-integral becomes [7]

$$Z = \int \mathcal{D}\lambda(t)\Delta_P(\lambda(0))\Delta_P(\lambda(T))\mathcal{D}\psi(x,t)\mathcal{D}\psi^\dagger(x,t)\exp[i(S_{YM} + S_F)] \tag{1.3}$$

where

$$\begin{aligned} &S_0 = \int_0^T dt\, \tfrac{L}{2}\textstyle\sum_a(\partial_t\lambda_a)^2 \\ &S_F = \int_0^T dt \int_0^L dx[\psi^\dagger i\partial_t\psi + \psi^\dagger\gamma^5(i\partial_x - g\lambda)\psi] + S_{coul} \\ &S_{coul} = -\textstyle\sum_{a,b}\int_0^T dt\int_0^L dx\int_0^L dy \rho_{ab}(x)\rho_{ba}(y)K_{ab}(x-y) \end{aligned} \tag{1.4}$$

The kernel $K_{ab}(x)$ is given by (for $x \in [-L, L]$)

$$\begin{aligned} &K_{ab}(x) = (g^2/4)e^{igx(\lambda_a - \lambda_b)}[L/(2\sin^2\tfrac{\pi(\lambda_a-\lambda_b)}{\lambda_0}) - ix\cot\tfrac{\pi(\lambda_a-\lambda_b)}{\lambda_0} - |x|],\ a \neq b \\ &K_{aa}(x) = (g^2/4)[L/6 - |x| + x^2/L] \end{aligned} \tag{1.5}$$

The action in (1.4) is invariant under the "large" gauge transformations [7, 8, 9]

$$\begin{aligned}&\psi_a(x,t) \to \Omega_{ab}(\vec{m},x)\psi_b(x,t), \quad \Omega_{ab}(\vec{m},x) \equiv \exp[-igxm_a\lambda_a]\\&\lambda_a(t) \to \lambda_a^{\Omega}(t) = \lambda_a(t) + m_a\lambda_0\end{aligned} \tag{1.6}$$

where

$$\lambda_0 \equiv \frac{2\pi}{gL} \tag{1.7}$$

The parameter λ_0 defines the range $[0, \lambda_0]$ of integration of the eigenvalues λ_a in (1.3). A simple way of understanding the period (1.7) is to consider the Wilson loop operators

$$W_m \equiv \frac{1}{N}\mathrm{Tr}U^m,\ U \equiv \exp(ig\int_0^L A_1 dx) \tag{1.8}$$

which in the gauge (1.2) evaluate to

$$W_m = \frac{1}{N}\sum_a \exp(igL\lambda_a) \equiv \frac{1}{N}\sum_a \exp(i2\pi\lambda_a/\lambda_0) \tag{1.9}$$

Mesonic operators [7] or H_F in (1.13) below, which involve gluons as well as quarks, are not periodic under $\lambda_a = 0 \to \lambda_a = \lambda_0$ *per se* but one needs to simultaneously transform the quarks also according to (1.6). This ultimately leads to a decompactification of eigenvalues in the effective theory of the gluons.

The factors $\Delta_P(\lambda(0))$ and $\Delta_P(\lambda(T))$ in the measure are defined by

$$\Delta_P(\lambda) \equiv \prod_{a<b} \sin[\pi(\lambda_a - \lambda_b)/\lambda_0] \tag{1.10}$$

These factors in the measure imply that the initial and final wavefunctions are completely antisymmetrized with respect to the λ_a's (since to start with they must be symmetric with repsect to permutation of the λ_a's on account of Weyl-symmetry). This gives rise to the well-known fermionic nature of these eigenvalues.

The hamiltonian corresponding to the action (1.4) is given by

$$H = H_{YM} + H_F, \tag{1.11}$$

where

$$H_{YM} = \frac{1}{2L} \sum_a p_a^2, \tag{1.12}$$

and

$$\begin{aligned} H_F &= H_{F,0} + H_{coul} \\ H_{F,0} &= \textstyle\sum_a \int_0^L dx\, \psi^{\dagger a} \gamma^5 (-i\partial_x + g\lambda_a) \psi^a \\ H_{coul} &= -\textstyle\sum_{ab} \int_0^L dx \int_0^L dy\, \rho_{ab}(x) \rho_{ba}(y) K_{ab}(x-y) \end{aligned} \tag{1.13}$$

The kernels $K_{ab}(x-y)$ are given by (1.5).

The physical Hilbert space of the above theory satisfies the zero charge condition (for each colour) [7]

$$Q_a \equiv \int dx \rho_{aa}(x) = 0 \tag{1.14}$$

2 Ground State

In this section we construct the ground state of the hamiltonian (1.11) in the $1/N$ expansion. We do it in two steps. (a) We first consider the gauge field as external and discuss the dynamics of the fermions for fixed λ_a's. We show that for any fixed λ_a's the dirac sea built out of free fermions is the ground state of this problem modulo $1/N^2$ corrections. (b) Next we use this result to construct the ground state of the full problem where both fermions and the eigenvalues are dynamical.

We should remark that the expectation value of H in the full ground state of the theory was already presented in [7]. In the discussion below we will allow some small overlap with [7] for the sake of completeness.

2.1 Fermion dynamics in an external background of gauge fields

We start by discussing eigenstates of $H_{F,0}$ (eqn. (1.13)) in the presence of fixed background values of $\{\lambda_a\}$. We will include the effect of H_{coul} later on. For simpilcity we will also work with $n_f = 1$ at first.

Since $H_{F,0}$ is quadratic, the ground state is simply given by filling the fermi sea according to the single-particle spectrum

$$E^a_{n\alpha} = \frac{2\pi}{L}\mathrm{sgn}(\alpha)(n + \frac{\lambda_a}{\lambda_0}) \tag{2.1}$$

Eqn. (2.1) is obtained by noting that the dirac equation $[i\partial_t + \gamma^5(i\partial_x - g\lambda_a)]\psi^a_\alpha = 0$ is solved by

$$\psi^a_\alpha(x,t) = \sum_n \exp[-iE^a_{\alpha,n}t + i\frac{2\pi n}{L}x]\psi^a_{\alpha,n} \tag{2.2}$$

with $E^a_{\alpha,n}$ as in (2.1). Here we have chosen the convention $\gamma^0 = \sigma_1$ and $\gamma^1 = -i\sigma_2$, so that $\gamma^5 \equiv \gamma^0\gamma^1 = \sigma_3$. $\alpha = \pm 1$ represent the right-, left-moving fermions respectively.

In filling the fermi sea, *a priori* the fermi levels could be different for different chirality and different colour. However, as we noted in [7], translation invariance of the vacuum demands that the right fermi-momentum must be left fermi-momentum minus one for each colour. Let us denote the right fermi-momentum for colour a as $p^a_F = 2\pi n^a_F/L$. The fermi (dirac) sea is therefore the state $|\vec{n_F}\rangle$ defined by

$$\begin{aligned} \psi^a_{R,n}|\vec{n_F}\rangle &= 0, n > n^a_F \\ \psi^{\dagger a}_{R,n}|\vec{n_F}\rangle &= 0, n \le n^a_F \\ \psi^a_{L,n}|\vec{n_F}\rangle &= 0, n \le n^a_F \\ \psi^{\dagger a}_{L,n}|\vec{n_F}\rangle &= 0, n > n^a_F \end{aligned} \tag{2.3}$$

The fermion modes $\psi^a_{\alpha,n}$ are defined by (2.2) and satisfy the anticommutation relation

$$\{\psi^a_{\alpha,n}, \psi^{\dagger b}_{\beta,m}\} = \frac{1}{L}\delta_{\alpha\beta}\delta_{mn} \tag{2.4}$$

The reader might wonder how the fermi levels can be different for different colours and still produce a colour-singlet state. Indeed the question is further complicated by the fact that the fermi levels n^a_F, having come from momentum labels of fermions, are non-gauge-invariant; under the large gauge transformation (1.6) they transform as

$$n^a_F \to (n^a_F)^\Omega = n^a_F - m_a \tag{2.5}$$

We will defer the detailed discussion of gauge-invariance of the vacuum till Sec. 2.2. For the moment, let us check that the state $|\vec{n_F}\rangle$ satisfies the constraint (1.14). Note that $Q_a = \sum_{\alpha=R,L} Q_{\alpha,a}$ where $Q_{\alpha,a} = L\sum_{m=-\infty}^{\infty} \psi^{\dagger a}_{\alpha,m}\psi^{a}_{\alpha,-m}$. Evaluating this on the state $|\vec{n_F}\rangle$ we get

$$Q^a_R = \sum_{m=-\infty}^{n^a_F} 1, \qquad Q^a_L = \sum_{m=n^a_F+1}^{\infty} 1 \tag{2.6}$$

The sums are divergent. By using the gauge-invariant exponential regulator [7] $\exp[-\epsilon|m+\lambda_a/\lambda_0|]$ we get

$$Q^a_R(\epsilon) = \frac{1}{\epsilon} + \frac{1}{2} + n^a_F + \frac{\lambda_a}{\lambda_0} + o(\epsilon), \quad Q^a_R(\epsilon) = \frac{1}{\epsilon} - \frac{1}{2} - n^a_F - \frac{\lambda_a}{\lambda_0} + o(\epsilon) \tag{2.7}$$

This leads to

$$Q_a(\epsilon) = Q^a_R(\epsilon) + Q^a_L(\epsilon) = \frac{2}{\epsilon} \tag{2.8}$$

This is a divergent c-number term which can be removed by normal ordering. Thus $:Q_a:= Q_a(\epsilon) - (2/\epsilon) = 0$. Note that we have been able to achieve this without assuming any special values of n^a_F.

Construction of ground state of H_F

It is easy to see that $|\vec{n_F}\rangle$ is an eigenstate of the dirac part $H_{F,0}$ of the hamiltonian (1.13)

$$H_{F,0}|\vec{n_F}\rangle = E_{F,0}|\vec{n_F}\rangle \tag{2.9}$$

$$E_{F,0} = g\lambda_0 \sum_a \sum_{m=-\infty}^{\infty} (m + \lambda_a/\lambda_0)\mathrm{tr}_d(\gamma^5 Q^a_m) \tag{2.10}$$

where

$$Q^a_{m,\alpha\beta} \equiv L\langle\vec{n_F}|\psi^{\dagger a}_{\alpha,m}\psi^a_{\beta,m}|\vec{n_F}\rangle = \theta(n^a_F - m)\frac{1+\gamma^5}{2} + \theta(m - n^a_F - 1)\frac{1-\gamma^5}{2} \tag{2.11}$$

Regarding the action of H_{coul} on $|\vec{n_F}\rangle$, it is more convenient to use the alternative form

$$H_{coul} = \sum_{\alpha,\beta} H_{\alpha\beta} \tag{2.12}$$

where

$$H_{\alpha\beta} = \frac{L}{2\lambda_0^2} \sum_{n=-\infty}^{\infty} \frac{\rho_{\alpha,ab,n}\rho_{\beta,ba,-n}}{[n + (\lambda_a - \lambda_b)/\lambda_0]^2} \tag{2.13}$$

In the above

$$\rho_{\alpha,ab,n} = \sum_{m=-\infty}^{\infty} \psi^{\dagger b}_{\alpha,m}\psi^{a}_{\alpha,m+n} \tag{2.14}$$

It is a lengthy but straightforward calculation to show that

$$[H_{RR} + H_{LL}]|\vec{n_F}\rangle = E_{coul}|\vec{n_F}\rangle \tag{2.15}$$

where

$$E_{coul} = \frac{g}{4\pi\lambda_0} \sum_{a,b} \sum_{m,m'=-\infty}^{\infty} \frac{\mathrm{tr}_d(Q^b_m Q^a_{m'})}{[m - m' + (\lambda_a - \lambda_b)/\lambda_0]^2} \tag{2.16}$$

where the Q^a_n have been introduced above in (2.11).

The left-right mixing terms $H_{LR} = H_{RL}$ take $|\vec{n_F}\rangle$ to an orthogonal state (which is a linear combination of states with two holes and two particles). If we treat H_{LR} as a perturbation term, the first order perturbation correction is zero since

$$\langle\vec{n_F}|H_{LR}|\vec{n_F}\rangle = 0 \tag{2.17}$$

Second order perturbation theory gives the contribution

$$E_{LR} = \frac{g}{8\pi^2\lambda_0^3} \sum_{a,b} \sum_{m=-\infty}^{\infty} \frac{m}{[m + n^a_F - n^b_F + (\lambda_a - \lambda_b)/\lambda_0]^4} \tag{2.18}$$

For finite N there is no reason to regard this as a perturbation. However in the large N limit we scale the coupling constant g as $g = \bar{g}/\sqrt{N}$ so that $\lambda_0 = \bar{\lambda}_0\sqrt{N}$. This makes E_{LR} down by $1/N$ compared to E_{coul} because of the two extra powers of λ_0 in the denominator. This is actually the story with conventional scaling of coupling constant. As we shall see in Sec. 3, the scaling in our theory also involves scaling of the length L of the circle. This in fact brings down E_{LR} by an additional factor of $1/N$.

Combining the above results, we find that the dirac sea is an eigenstate of H_F to the leading order in $1/N$, with the energy given by

$$H_F|\vec{n_F}\rangle = E_F|\vec{n_F}\rangle + o(1/N), \quad E_F = E_{F,0} + E_{coul} \tag{2.19}$$

The sums in (2.10) and (2.16) are divergent. Using the exponential regulator once again as in (2.7) we get

$$E_{F,0}(\epsilon) = g\lambda_0 \sum_a [-\frac{2}{\epsilon} - \frac{1}{12} + V_{\rm reg}(\xi_a) + o(\epsilon)] \tag{2.20}$$

and

$$E_{coul}(\epsilon) = \frac{g}{4\pi\lambda_0} \sum_{a,b} [\frac{\pi^2}{\epsilon \sin^2 \frac{\pi}{\lambda_0}(\xi_a - \xi_b)} + 2(\ln\epsilon - 1) + K(\frac{\xi_a - \xi_b}{\lambda_0}) + o(\epsilon \ln \epsilon)] \tag{2.21}$$

where

$$V_{\rm reg}(\xi_a) = (\frac{\xi_a}{\lambda_0} + \frac{1}{2})^2 \tag{2.22}$$

$$K_{\rm reg}(\frac{\xi_a - \xi_b}{\lambda_0}) = [2C + \psi(w_{ab}) + \psi(-w_{ab}) + w_{ab}\{\psi'(w_{ab}) - \psi'(-w_{ab})\}] \tag{2.23}$$

Here $w_{ab} \equiv (\xi_a - \xi_b)/\lambda_0$ and C is Euler's constant. Also, $\psi(x) = (d/dx)\ln\Gamma(x)$ and $\psi'(x) = \partial_x \psi(x)$, $\Gamma(x)$ being the standard gamma-function. Note that E_F only depends on the gauge-invariant combination

$$\xi_a \equiv \lambda_a + n_F^a \lambda_0 \tag{2.24}$$

The appearance of this variable is responsible for decompactification of the eigenvalue λ_a. We will discuss this in more detail shortly.

A remark is in order here justifying our definition of the regularized quantities $V_{\rm reg}$ and $K_{\rm reg}$. The divergent piece in $E_{F,0}(\epsilon)$ is a constant and, because of the constraint that the total number of eigenvalues is N, does not affect the dynamics. In the grand canonical ensemble the equivalent statement is that such a divergence can be cancelled by a simple additive renormalization of the chemical potential term. Similar remarks can be made about the $\ln(\epsilon)$ piece in $E_{coul}(\epsilon)$ and the constant finite pieces in $E_{F,0}(\epsilon)$ and $E_{coul}(\epsilon)$. The justification for ignoring the $1/(\epsilon \sin^2)$ piece in $E_{coul}(\epsilon)$ is more subtle. The main point is that this $1/\epsilon$ comes multiplied with the quadratic pole $(\xi_a - \xi_b)^{-2}$ which is the only singularity of (2.21) in the limit $\xi_a \to \xi_b$. If we take the same limit in the unregulated expression (2.16) we find a quadratic pole with residue $\sum_n (Q_n^a)^2 = \sum_n Q_n^a = Q_a$. Here we have used $(Q_n^a)^2 = Q_n^a \, \forall a, n$. Thus the $1/\epsilon$ in (2.21) can be identified with the total charge which must vanish by (1.14).

Combining all this we get the following regularized expression for the eigenvalue E_F

$$E_F = g\lambda_0 \sum_a (\xi_a/\lambda_0 + 1/2)^2 + \frac{g}{4\pi\lambda_0} \sum_{a,b} K_{\rm reg}(\frac{\xi_a - \xi_b}{\lambda_0}) \tag{2.25}$$

where $K_{\rm reg}$ is defined in (2.23).

In the above we have proved that the filled fermi sea is an eigenstate of H_F at leading $1/N$ order. How does one argue that it is actually the *ground* state of the system? If we ignore for the moment H_{coul} it is obvious that the filled fermi sea $|\vec{n_F}\rangle$ is the ground state of $H_{F,0}$ (for any given set of n_F^a and λ_a, or equivalently, for any given set of ξ_a's, which is probably a more appropriate specification of external background in our problem). In presence of H_{coul} the argument is not so simple. Let us present several independent reasons why the filled fermi sea should be the ground state. (a) In [7] we have shown that the expectation value of the meson bilocal operator $\overline{M}_{xy}(\lambda, t)$ (see eq. (55) of [7]) in this state provides the unique lowest energy translation invariant solution to the classical equation of motion. (b) The four-fermi interaction represented by H_{coul} is repulsive in nature. (c) It is easy to show that small number of gauge-invariant mesonic excitations always *increase* the energy of this state. (d) In the scaling that we describe in the next section (involving g and L) H_{coul} is subleading to $H_{F,0}$ by a factor of $1/N$. Thus, to leading order in $1/N$ the filled fermi sea must be the ground state.

2.2 Effective hamiltonian for gauge fields and ground state of the full theory

Now that we have computed the ground state of H_F for fixed external gauge field, let us construct the full ground state by the method of separation of variables. What we will do now is similar in spirit to solving the Schrodinger problem for a central force in three dimensions where we look for solutions which are products of radial and angular wave-functions. We solve the angular problem first and find the eigenfunctions of the angular momentum operator. The centrifugal term L^2/r is evaluated by using the eigenvalue of the angular momentum at a fixed r. This is then put back in the full laplacian to derive an effective hamiltonian for the radial problem which is then solved by appropriate methods.

The schematic correspondence between the above example and our problem will be $\psi \leftrightarrow (\theta,\phi)$, $\vec{\lambda} \leftrightarrow r$, $H_F[\psi,\psi^\dagger,\vec{\lambda}] \leftrightarrow L^2(\theta,\partial_\theta,\partial_\phi)/r$, $|\vec{n_F}\rangle \leftrightarrow Y_{0,0}(\theta,\phi)$ and $H_{YM} \leftrightarrow -(1/r^2)\partial_r(r^2\partial_r)$. Note the important fact that $|\vec{n_F}\rangle$ is independent of the λ_a's as is clear from the defintion (2.3).

Let us write a product wavefunction of the system as

$$|\Psi\rangle = |\vec{n_F}\rangle \otimes \Phi(\vec{\lambda}) \tag{2.26}$$

Using (1.11), (2.19) and (2.25) we find that

$$H|\Psi\rangle = |\vec{n_F}\rangle \otimes H_{\text{eff}}\Phi(\vec{\lambda}) \tag{2.27}$$

where

$$H_{\text{eff}} = \sum_a \frac{g\lambda_0}{4\pi}(-\partial^2_{\lambda_a}) + E_F \tag{2.28}$$

with E_F given by (2.25).

Our first guess at the full ground state would be $|\Psi\rangle = |\vec{n_F}\rangle \otimes \Phi_0(\vec{\lambda})$ where $\Phi_0(\vec{\lambda})$ is the ground state of H_{eff}. However, as we have discussed in great detail in [7] such a state $|\Psi\rangle$ is not gauge-invariant because of the shift (2.5) of the fermi levels under large gauge transformations. The correct gauge-invariant ground state is given by a sum over fermi levels [7]

$$|\Psi_0\rangle = \sum_{\vec{n}_F} |\vec{n_F}\rangle \otimes \Phi^{(0)}_{\vec{n}_F}(\vec{\lambda}) \tag{2.29}$$

where the wavefunctions $\Phi^{(0)}_{\vec{n}_F}(\vec{\lambda})$ are given by

$$\Phi^{(0)}_{\vec{n}_F}(\vec{\lambda}) = u^{(0)}(\vec{\lambda} + \vec{n}_F) \tag{2.30}$$

Here $u^{(0)}(\vec{\xi})$, $\xi \in (-\infty,\infty)$ is the ground state wave-function for the hamiltonian (2.28) where in the kinetic term the operator ∂_{λ_a} has been replaced by ∂_{ξ_a}. In other words, $u^{(0)}(\vec{\xi})$ satisfies the differential equation

$$\mathcal{H}u^{(0)}(\vec{\xi}) = E_0 u^{(0)}(\vec{\xi}) \tag{2.31}$$

where

$$\mathcal{H} = \sum_a \frac{g\lambda_0}{4\pi}(-\partial^2_{\xi_a}) + E_F(\vec{\xi}) \tag{2.32}$$

In the above E_0 denotes the lowest eigenvalue of the hamiltonian. We shall discuss in the next section how to evaluate it.

Note that in expectation values computed in the full theory, the implication of the sum-of-product structure (2.29) of the full wave-function is that the effective range of the eigenvalues becomes the real line. Let us calculate, for instance, the expectation value of the full hamiltonian H in the state $|\Psi_0\rangle$.

$$\langle\Psi_0|H|\Psi_0\rangle = \sum_{\vec{n}_F}\prod_a \int_0^{\lambda_0} d\lambda_a \, \Phi^{(0)*}_{\vec{n}_F}(\vec{\lambda}) H_{\text{eff}} \Phi^{(0)}_{\vec{n}_F}(\vec{\lambda}) = \prod_a \int_{-\infty}^{\infty} d\xi_a \, u^{(0)*}(\vec{\xi})\mathcal{H}u^{(0)}(\vec{\xi}) \tag{2.33}$$

where H_{eff} and $\mathcal{H}$ are given by (2.28) and (2.32). The last line shows that the presence of fermions forces a *decompactification* of the eigenvalues λ_a to the gauge-invariant combination $\lambda_a + n_F^a\lambda_0 = \xi_a$ which is a real number $\in (-\infty,\infty)$. It also tells us that there is no fixed fermi level in a compact gauge theory with dynamical gauge fields; rather, the effective theory of the gauge fields $\vec{\lambda}$ in the fermi vacuum is given by a density matrix $\rho(\vec{n}_F,\vec{\lambda}|\vec{n'_F},\vec{\lambda'})$. The sum over $\vec{n_F}$ in the above equation corresponds to taking trace over this density matrix.

3 The Large-N Expansion

In this section we will discuss the large N limit in detail and present the $1/N$ expansion for some physical quantities. For concreteness, we will consider the partition function

$$\exp(-\beta F) \equiv Z = \mathrm{Tr}\exp(-\beta H) \tag{3.1}$$

where $H = H_{YM} + H_F$ is given by (1.11). We will also evaluate towards the end of this section the vacuum expectation value of the Wilson loop operator $\mathrm{Tr}U^m$ defined by (1.8).

The large N limit involves defining a scaled coupling constant

$$\bar{g} = g\sqrt{N} \tag{3.2}$$

which is held fixed as $N \to \infty$. According to (1.7), we must also define a scaled eigenvalue-period

$$\lambda_0 = \sqrt{N}\bar{\lambda}_0, \ \bar{\lambda}_0 = \frac{2\pi}{\bar{g}L} \tag{3.3}$$

This necessitates scaling of $\bar{\lambda}_a, \xi_a$ and p_a to

$$\bar{\lambda}_a = \lambda_a/\sqrt{N},\ \bar{\xi}_a = \xi_a/\sqrt{N},\ \bar{p}_a = p_a/\sqrt{N} \tag{3.4}$$

Note that in terms of the scaled variables $\bar{p}_a = -(i/N)\partial_{\bar{\lambda}_a}$, implying that $\hbar$ is $1/N$. Let us, for example, rewrite Eqns. (2.31) and (2.28) for the ground state energy and eigenfunctions in terms of the scaled variables

$$\mathcal{H} u^{(0)}(\{\bar{\xi}_a\}) = E_0 u^{(0)}(\{\bar{\xi}_a\}) \tag{3.5}$$

$$\begin{aligned}
&\mathcal{H} = \mathcal{H}_{YM} + \mathcal{H}_1 + \mathcal{H}_2 \\
&\mathcal{H}_{YM} = (\bar{g}\bar{\lambda}_0/4N\pi) \textstyle\sum_a (-\partial^2_{\bar{\xi}_a}) \\
&\mathcal{H}_1 = \bar{g}\lambda_0 \textstyle\sum_a (\bar{\xi}_a/\lambda_0 + 1/2)^2 \\
&\mathcal{H}_2 = (\bar{g}/4\pi N\bar{\lambda}_0) \textstyle\sum_{a,b} K_{\rm reg}([\bar{\xi}_a - \bar{\xi}_b]/\bar{\lambda}_0)
\end{aligned} \tag{3.6}$$

where the $K_{\rm reg}$ has been defined in (2.23) (note that in (2.23) w_{ab} is also equal to $[\bar{\xi}_a - \bar{\xi}_b]/\bar{\lambda}_0$).

3.1 $\beta \to \infty$ limit and the ground state energy

In the $\beta \to \infty$ limit, the only state contributing to the trace in (3.1) is the ground state, discussed in the previous section. Thus the free energy F simply coincides with the ground state energy E_0 of (3.5).

Let us now discuss how to determine E_0. If the two-body interaction $\mathcal{H}_2$ was absent, E_0 would be simply given by

$$E_{0,0} = \frac{\bar{g}}{2\sqrt{\pi}} N^{3/2} \tag{3.7}$$

Proof: The ground state for the N-fermion hamiltonian $\mathcal{H}_{YM} + \mathcal{H}_1$ is given by the Slater determinant

$$u_0^0(\bar{\xi}) = \frac{1}{\sqrt{N!}} {\rm Det}_{i,j=0}^{N-1} \phi_i(\bar{\xi}_{j+1}) \tag{3.8}$$

where $\phi_n(\bar{\xi}), n = 0, \ldots, \infty$ are normalized single-particle wavefunctions satisfying

$$\frac{\bar{g}\bar{\lambda}_0}{2\pi}[-\frac{\partial^2_{\bar{\xi}}}{2N} + 2\pi(\frac{\bar{\xi}}{\bar{\lambda}_0} + \frac{1}{2})^2]\phi_n(\bar{\xi}) = \epsilon_n \phi_n(\bar{\xi}) \tag{3.9}$$

It is easy to find explicit solutions of the above equation:

$$\phi_n(\bar{\xi}) = (\frac{2^n n!}{\bar{\alpha}\sqrt{N\omega}})^{-1/2} H_n(\sqrt{\omega}x)\exp(-\omega x^2/2),\ \omega^2 \equiv 4\pi \tag{3.10}$$

where the variable x is defined by

$$(\bar{\xi}/\bar{\lambda}_0) + (1/2) = \bar{\alpha}x,\ \bar{\alpha} \equiv \bar{\lambda}_0^{-1/2} N^{-1/4} \tag{3.11}$$

In the above H_n are the standard Hermite polynomials. The energy eigenvalues ϵ_n are given by

$$\epsilon_n = \frac{\bar{g}}{\sqrt{\pi N}}(n + \frac{1}{2}) \quad n = 0, 1, \ldots \tag{3.12}$$

The ground state energy $E_{0,0}$ which is a sum over the first N energy levels clearly reproduces (3.7).

For later use let us also evaluate the density operator

$$\rho(\bar{\xi}) = \frac{1}{N}\sum_{i=1}^{N}\delta(\bar{\xi} - \bar{\xi}_a) \tag{3.13}$$

in the state (3.8). The result is [10]

$$\rho_{0,0}(\bar{\xi}) = \tfrac{1}{N}\textstyle\sum_{n=0}^{N-1}|\phi_n(\bar{\xi})|^2 = 2\bar{\alpha}\pi^{-3/4}[(1 - x^2\tfrac{\sqrt{\pi}}{N})^{1/2}\theta(1 - x^2\tfrac{\sqrt{\pi}}{N}) + o(1/N)] \tag{3.14}$$

So far we have ignored the two-body interaction $\mathcal{H}_2$ in (3.6) (the additional subscript 0 on the ground state wavefunction and energy denotes this fact). We will see shortly that in the only sensible scaling available in the theory this term will be of lower order in N. In other words, the correction terms to (3.7) coming from the two-body interaction will turn out to be $1/N$ lower order than $N^{3/2}$, leading to the result that the total ground state energy is $E_0 = N^{3/2}\bar{g}/(2\sqrt{\pi}) + o(N^{1/2})$.

Collective field theory

Since the result $E_0 \propto N^{3/2}$ is rather unexpected (naively one would expect the pure Yang-Mills result N^2), let us try to understand this from some simple scaling analysis. The easiest framework to do such an analysis is collective

field theory [11]. The hamiltonian $\mathcal{H}$ in (3.6) corresponds to the following collective field theory hamiltonian (the subscript 'c' stands for collective)

$$\begin{aligned}
\mathcal{H}_c &= \mathcal{H}_{cYM} + \mathcal{H}_{c1} + \mathcal{H}_{c2} \\
\mathcal{H}_{cYM} &= N^2(\bar{g}\bar{\lambda}_0/2\pi)\ \textstyle\int d\bar{\xi}\,(\rho(\bar{\xi})/2)[\Pi^2(\bar{\xi}) + \frac{\pi^2}{3}\rho^2(\bar{\xi})] \\
\mathcal{H}_{c1} &= N(\bar{g}\bar{\lambda}_0/2\pi)\ \textstyle\int d\bar{\xi}\,\rho(\bar{\xi})[2\pi(\bar{\xi}/\bar{\lambda}_0 + 1/2)^2] \\
\mathcal{H}_{c2} &= N(\bar{g}\bar{\lambda}_0/2\pi)\ (1/2\bar{\lambda}_0^2) \textstyle\int d\bar{\xi} \int d\bar{\xi}'\, K_{\text{reg}}(\bar{\xi} - \bar{\xi}'/\bar{\lambda}_0)\rho(\bar{\xi})\rho(\bar{\xi}')
\end{aligned} \tag{3.15}$$

Here $\rho(\bar{\xi})$ is the eigenvalue density defined in (3.13). $\Pi(\bar{\xi})$ is defined such that $\Pi(\bar{\xi})\rho(\bar{\xi})$ corresponds to the momentum density: $\int d\bar{\xi}\,\Pi(\bar{\xi})\rho(\bar{\xi}) = (1/N)\sum_a \bar{p}_a$. This implies the following commutation relation

$$[\rho(\bar{\xi}), \Pi(\bar{\xi}')] = -\frac{i}{N^2}\partial_{\bar{\xi}}\delta(\bar{\xi} - \bar{\xi}') \tag{3.16}$$

The theory is defined with the constraint

$$\int d\bar{\xi}\rho(\bar{\xi}) = 1 \tag{3.17}$$

which is a consequence of the definition (3.13).

Classical analysis

Let us regard the collective field hamiltonian $\mathcal{H}_c$ as a function of the classical variables $\rho(\bar{\xi}), \Pi(\bar{\xi})$. The Poisson bracket between them is simply (3.16) without the i on the right hand side

$$[\rho(\bar{\xi}), \Pi(\bar{\xi}')]_{PB} = -\frac{1}{N^2}\partial_{\bar{\xi}}\delta(\bar{\xi} - \bar{\xi}') \tag{3.18}$$

This leads to the following equations of motion

$$\begin{aligned}
\partial_t\rho(\bar{\xi}) &= -\tfrac{\bar{g}\bar{\lambda}_0}{2\pi}\partial_{\bar{\xi}}(\rho(\bar{\xi})\Pi(\bar{\xi})) \\
\partial_t\Pi(\bar{\xi}) &= -\tfrac{\bar{g}\bar{\lambda}_0}{2\pi}\partial_{\bar{\xi}}[\tfrac{\Pi^2}{2} + \tfrac{\pi^2}{2}\rho^2 + \tfrac{2\pi}{N}(\bar{\xi}/\bar{\lambda}_0 + 1/2)^2 + \\
&\qquad\qquad \tfrac{1}{N\bar{\lambda}_0^2}\textstyle\int d\bar{\xi}'\,\rho(\bar{\xi}')K_{\text{reg}}((\bar{\xi} - \bar{\xi}')/\bar{\lambda}_0)]
\end{aligned} \tag{3.19}$$

For time-independent solutions these reduce to

$$\partial_{\bar{\xi}}[\frac{\pi^2}{2}\rho^2 + \frac{2\pi}{N}(\bar{\xi}/\bar{\lambda}_0 + 1/2)^2 + \frac{1}{N\bar{\lambda}_0^2}\int d\bar{\xi}'\,\rho(\bar{\xi}')K_{\rm reg}((\bar{\xi}-\bar{\xi}')/\bar{\lambda}_0)] = 0 \quad (3.20)$$

If we take the naive $N \to \infty$ limit, the potential and interaction terms drop out and we get a constant density

$$\rho(\bar{\xi}) = \text{constant} \quad (3.21)$$

However, since that the range of $\bar{\xi}$ is noncompact (due to the phenomenon of decompactification discussed in the last section) such a constant density is unnormalizable, that is, it cannot satisfy (3.17). There are no gauge-invariant cutoffs available on the range of $\bar{\xi}$ either which can save the situation.

Clearly in order to obtain normalizable solutions for $\rho(\bar{\xi})$ the large N limit must be taken in such a way that one or both of the terms $\mathcal{H}_{c1}$ and $\mathcal{H}_{c2}$ in (3.15) are of the same order as $\mathcal{H}_{cYM}$. Now it is easy to see that only $\mathcal{H}_{c2}$ by itself, being translation invariant, cannot produce localization. Indeed, if one includes in (3.20) terms coming from $\mathcal{H}_{cYM}$ and $\mathcal{H}_{c2}$ and drops those coming from $\mathcal{H}_{c1}$, the solution is still $\rho(\bar{\xi})$ = constant, which is untenable. Thus we must ensure that the simple harmonic potential term in (3.20) survives in the large N limit (irrespective of what happens to the interaction term). In other words, in (3.15) $\mathcal{H}_{cYM}$ and $\mathcal{H}_{c1}$ must be of the same order.

The clue to how the above can be achieved is provided by the formula (3.14). This suggests that the correct coordinate in terms of which a sensible N scaling of the collective field theory may be available is not $\bar{\xi}$, but y, defined by

$$x = \sqrt{N}y, \quad \text{equivalently} \quad \frac{\bar{\xi}}{\bar{\lambda}_0} + \frac{1}{2} = \alpha y, \quad (3.22)$$

where

$$\alpha = \bar{\lambda}_0^{-1/2} N^{1/4} \equiv \bar{\alpha}\sqrt{N} \quad (3.23)$$

The new density variable $\widetilde{\rho}(y)$ is given by

$$\widetilde{\rho}(y) = \rho(\bar{\xi})d\bar{\xi}/dy = \rho(\bar{\xi})N^{1/2}\alpha^{-1} \quad (3.24)$$

Note that the density expectation value (3.14) in terms of the new variable reduces to

$$\widetilde{\rho}_{0,0}(y) \equiv \rho_{0,0}(\bar{\xi})N^{1/2}\alpha^{-1} = 2\pi^{-3/4}(1-y^2\sqrt{\pi})^{1/2}\theta(1-y^2\sqrt{\pi}) \quad (3.25)$$

which is an N-free expression. We also define a new momentum variable $\widetilde{\Pi}(y)$ by

$$\widetilde{\Pi}(y) = \Pi(\bar{\xi}) N^{1/2} \alpha^{-1} \tag{3.26}$$

such that the Poisson bracket is kept invariant. This also ensures that N scales out of the combination $\Pi^2(\bar{\xi}) + \rho^2(\bar{\xi})/12$ in $\mathcal{H}_{cYM}$, Eqn. (3.15). In terms of these, the collective field hamiltonian becomes

$$\begin{aligned}
\mathcal{H}_{cYM} &= N^{3/2}\,(\bar{g}/2\pi)\ \int dy\,(\widetilde{\rho}(y)/2)[\widetilde{\Pi}^2(y) + (\pi^2/3)\widetilde{\rho}^2(y)] \\
\mathcal{H}_{c1} &= N^{3/2}\,(\bar{g}/2\pi)\ \int dy\,\widetilde{\rho}(y)\,2\pi y^2 \\
\mathcal{H}_{c2} &= N^{1/2}\,(\bar{g}/2\pi)\alpha^2\ \int dy\ \int dy'\,\widetilde{\rho}(y)\widetilde{\rho}(y')K_{\mathrm{reg}}(\alpha(y-y'))
\end{aligned} \tag{3.27}$$

There are several remarks to be made here. First of all, note that $\mathcal{H}_{cYM}$ and $\mathcal{H}_{c1}$ scale as $N^{3/2}$. It is easy to show that if we ignore the $\mathcal{H}_{c2}$ piece, then the time-independent equations (recall (3.20)) in the new variables have a solution $\widetilde{\rho}(y)$ which exactly coincides with (3.25) and the the classical energy of this configuration is identical to (3.7). The second point is, $\mathcal{H}_{c2}$ is subleading compared to the first two terms in $\mathcal{H}_c$ by $1/N$, if we demand that α is N-independent as $N \to \infty$. One can therefore treat this term as a perturbation. The reason for demanding N-independence of α is that that is the only way a scaled form of $\mathcal{H}_{c2}$ can be obtained. Indeed one can easily rule out N-dependence of α on other physical grounds also. If, for example, α grew with N, the function $K_{\mathrm{reg}}(\alpha(y-y'))$ would become infinitely discontinous (its poles become infinitely dense). If, on the other hand, α went as some inverse power $N^{-\gamma}, \gamma > 0$, then by using the formula

$$K_{\mathrm{reg}}(\alpha y) = \sum_{m=1}^{\infty} \zeta(2m+1)(\alpha y)^{2m} \tag{3.28}$$

one can see that although from the point of view of convergence the $\mathcal{H}_{c2}$ perturbation series would be sensible, it would have the unphysical feature that different parts of the two-body interaction contribute in different orders of perturbation theory leading to an incorrect representation of the nature of the interaction.

Scaling of length

Eqn. (3.23) is equivalent to

$$L = \frac{\alpha^2 2\pi}{\bar{g}} N^{-1/2} \tag{3.29}$$

In view of the fact that α is N-independent, we see that as $N \to \infty$, the (bare) length L of the space circle goes to zero as $L = N^{-1/2}\bar{L}$ where $\bar{L}$ is held constant. The parameter α can be identified as $\sqrt{\bar{g}\bar{L}/2\pi}$.

Scaling of time and energy

Let us rewrite the classical equations of motion using the variables $\widetilde{\rho}(y)$, $\widetilde{\Pi}(y)$. We get

$$\begin{aligned} &N^{1/2}\,\partial_t \widetilde{\rho}(y) = -(\bar{g}/2\pi)\partial_y(\widetilde{\rho}(y)\widetilde{\Pi}(y)) \\ &N^{1/2}\,\partial_t \widetilde{\Pi}(y) = -(\bar{g}/2\pi)\partial_y[\tfrac{\widetilde{\Pi}^2}{2} + \tfrac{\pi^2}{2}\widetilde{\rho}^2 + 2\pi y^2 + \tfrac{\alpha^2}{N}\textstyle\int dy'\,\widetilde{\rho}(y')K_{\rm reg}(\alpha(y-y'))] \end{aligned} \tag{3.30}$$

It is clear that N scales out of the classical equations if we work in terms of a scaled time $\bar{t}$ defined by

$$t = N^{1/2}\bar{t} \tag{3.31}$$

Now a scaling of time (in the Euclidean framework a scaling of β) implies an inverse scaling of energy

$$E = N^{-1/2}\bar{E} \tag{3.32}$$

In other words, $\exp(iHt) = \exp(i\bar{H}\bar{t}), \exp(-\beta H) = \exp(-\bar{\beta}\bar{H})$.

In this new scaling, the collective field hamitonian becomes

$$\bar{\mathcal{H}}_c = \frac{N^2\bar{g}}{2\pi}\int dy\,\frac{\widetilde{\rho}(y)}{2}[\widetilde{\Pi}^2 + \frac{\pi^2}{3}\widetilde{\rho}^2 + 4\pi y^2 + \frac{\alpha^2}{N}\int dy'\,\widetilde{\rho}(y')K_{\rm reg}(\alpha(y-y'))] \tag{3.33}$$

The scaled ground state energy is, therefore,

$$\bar{E_0} = N^2\frac{\bar{g}}{2\sqrt{\pi}}[1 + o(1/N)] \tag{3.34}$$

In the following we will study how the $1/N$ corrections follow from a WKB analysis.

$1/N$ corrections and WKB

The commutation relation between $\widetilde{\rho}(y)$ and $\widetilde{\Pi}(y)$ is given by (recall (3.16))

$$[\widetilde{\rho}(y), \widetilde{\Pi}(y')] = -\frac{i}{N^2}\partial_y \delta(y-y') \tag{3.35}$$

which implies that $\widetilde{\Pi}(y)$ can be represented by the following differential operator representation

$$\widetilde{\Pi}(y) = -\frac{i}{N^2}\partial_y \frac{\delta}{\delta\widetilde{\rho}(y)} \tag{3.36}$$

To construct a WKB solution we consider wave-functions of the form

$$\Psi[\widetilde{\rho}] = \exp\left(iN^p S[\widetilde{\rho}]\right) \tag{3.37}$$

where we have kept the real number p unspecified and will determine it by demanding a scaled time-independent Schrodinger equation. The latter is given by

$$\bar{\mathcal{H}_c}\Psi[\widetilde{\rho}] = \bar{E}\Psi[\widetilde{\rho}] \tag{3.38}$$

where $\bar{\mathcal{H}_c}$ is given by (3.33). Using the above differential operator representation of $\widetilde{\Pi}(y)$, (3.36), we get the following equation determining $S[\widetilde{\rho}]$ in terms of $\bar{E}$:

$$\begin{aligned}\bar{E} = \tfrac{N^2\bar{g}}{2\pi}\int dy \tfrac{\widetilde{\rho}}{2}[N^{2p-4}(\partial_y \tfrac{\delta S}{\delta\widetilde{\rho}(y)})^2 - iN^{p-4}(\partial_y \tfrac{\delta}{\delta\widetilde{\rho}(y)})^2 S + \tfrac{\pi^2}{3}\widetilde{\rho}^2 + 4\pi y^2 + \\ \tfrac{\alpha^2}{N}\int dy' \widetilde{\rho}(y') K_{\mathrm{reg}}(\alpha(y-y'))]\end{aligned} \tag{3.39}$$

In order to get a scaled Hamilton-Jacobi equation in the leading N order we must put $p = 2$. This implies the following perturbation series for S

$$\Psi[\widetilde{\rho}] = \exp[iN^2 S], \qquad S = S_0 + \frac{1}{N}S_1 + \frac{1}{N^2}S_2 + \ldots \tag{3.40}$$

It is easy to write down from (3.39) equations determining S_n in terms of the lower order terms.

The above analysis also implies that the energy $\bar{E}_0$ for the ground state should receive corrections to (3.7) as follows

$$\bar{E}_0 = N^2\Big[\frac{\bar{g}}{2\sqrt{\pi}} + \frac{1}{N}\Delta_{0,1} + \frac{1}{N^2}\Delta_{0,2} + \ldots\Big] \tag{3.41}$$

We have explicitly verified this by computing the many-body perturbation theory diagrams of the hamiltonian (3.15).

Introduction of multiple flavours

In the above we have considered the case of a single flavour, $n_F = 1$. It is easy to extend the above calculations to multiple flavours. The essential new physics point is the following. The dirac sea $|F\rangle$ for $n_F > 1$ flavours of free fermions is given by

$$\begin{aligned} \psi^a_{i,n}|F\rangle &= 0 \text{ for } n > n_F^{a,i} \\ \psi^{a\dagger}_{i,n}|F\rangle &= 0 \text{ for } n \le n_F^{a,i} \end{aligned} \tag{3.42}$$

where ψ's above represent the right-handed fermions. The left-handed fermions represent similar equations corresponding to the filling of all the levels starting from $n_F^{a,i} + 1$ *upwards*.

If the fermi levels for different flavours are different, then one would anticipate several conceptual difficulties. For instance, if there is no notion of a single n_F^a how does one construct $\bar{\xi}_a = \lambda + n_F^a \bar{\lambda}_0$ to describe the decompactified effective theory of the eigenvalues? Fortunately, because of the chiral $U(n_f) \times U(n_f)$ symmetry of the hamiltonian, we must demand that the ground state is a flavour singlet. This happens only if the $n_F^{a,i}$s above are all independent of i (this can be proved by operating the flavour rotation generators on $|F\rangle$). Having said that, we can again use the notation $|\vec{n_F}\rangle$ instead of $|F\rangle$ to represent the dirac sea.

Using the above, it is easy to show that the dirac sea is again an eigenstate of the hamiltonian upto $1/N$ corrections. In other words (2.19) is again valid, except that the energy eigenvalue E_F is now n_f times the expression (2.25). As a result in the calculation of the ground state energy according to (3.6), $\mathcal{H}_{cYM}$ remains the same but $\mathcal{H}_{c1}$ and $\mathcal{H}_{c2}$ get multiplied by n_f. It is simple to show that the ground state energy is now given by

$$\bar{E}_0 = N^2[\sqrt{n_f}\frac{\bar{g}}{2\sqrt{\pi}} + o(1/N)] \tag{3.43}$$

The eigenvalue density is given by

$$\widetilde{\rho}(y) = 2n_f^{1/4}\pi^{-3/4}(1 - y^2\sqrt{\pi n_f})^{1/2}\theta(1 - y^2\sqrt{\pi n_f}) + o(\frac{1}{N}) \tag{3.44}$$

Eqn. (3.43) clearly shows that adding fermions affects the leading N behaviour of the vacuum energy.

3.2 Finite β partition function and Excitations

The method presented above can be generalized to $\beta \neq 0$. In this case, one needs to consider excited states in the fermionic theory. This leads to subleading corrections (in $1/N$) to the effective hamiltonian for the gauge fields. The free energy can again be computed. We will not present here the explicit result but merely state that the N-scalings presented above also work at finite values of β and that the leading term (in N) in the expression for the free energy still has a non-trivial dependence on n_f.

3.3 Vacuum expectation value of the Wilson loop operator

In order to understand how the introduction of dynamical fermions affects the leading large N behaviour of various quantities, it is instructive to calculate vacuum expectation value $\langle W_m \rangle = \langle \mathrm{Tr} U^m \rangle$ of the Wilson loop operator (recall (1.8) and (1.9)). Using the expression for the ground state wavefunction (2.29) and the definition (3.13) of the density operator it is easy to show that

$$\langle W_m \rangle = \int d\bar{\xi} \langle \rho(\bar{\xi}) \rangle e^{i2\pi m \bar{\xi}/\bar{\lambda}_0} \tag{3.45}$$

On the right hand side the expectation value is meant in the ground state of the effective theory of eigenvalues defined by (3.5) and (3.6), and is given by (3.44). Using (3.22) to express the exponent in terms of y we get

$$\langle W_m \rangle = 2(-1)^m (1 + \frac{\partial^2}{\partial {x_m}^2}) J_0(x_m) + o(1/N), \quad x_m \equiv 2\pi^{3/4} \alpha \frac{m}{n_f^{1/4}} \tag{3.46}$$

where J_0 denotes the Bessel function of order zero.

Let us compare this with the result in pure Yang-Mills theory on a two-dimensional cylinder. The latter theory is described by the path integral (1.3) with no fermion integration and $S_F = 0$. It is easy to show in this case the eigenvalue density is a constant (which, unlike with fermions, is perfectly

allowed because the range of eigenvalues remains compact) and therefore

$$\langle W_m \rangle_{YM} = \delta_{n,0} + o(\frac{1}{N}) \tag{3.47}$$

One can also recover this by taking $n_f \to 0$ in (3.46).

It is clear that the introduction of fermions changes the result for $\langle \mathrm{Tr} U^m \rangle$ in the leading N order.

4 Concluding Remarks

To conclude we remark that QCD_2 on the cylinder shows considerable amount of subtlety and surprise. The main surprise is that the leading term in a $1/N$ expansion for the vacuum energy and for the *vev* of the Wilson loop operator is changed by the existence of fermions, contrary to what one would expect by standard N-counting arguments. The essential reason for this is that coupling to quarks leads to a 'decompactification' of the eigenvalues. This effect is not perturbative and persists even at $N = \infty$, thus changing the leading behaviour of pure Yang-Mills theory. Indeed it is a very interesting question what the present analysis can tell us about $3+1$ dimensional gauge theories in a compact space. In particular it would be interesting to know whether the phenonmenon of decompactification persists in four dimensions and whether as a result the pure Yang-Mills theory is affected by inclusion of quarks in the leading N order.

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